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Enhanced residual-free bubble method for convection–diffusion problems

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SUMMARY

We analyse the performance of the *enhanced residual-free bubble* (RFBe) method for the solution of elliptic convection-dominated convection–diffusion problems in 2-D, and compare the present method with the standard residual-free bubble (RFB) method. The advantages of the RFBe method are two-fold: it has better stability properties and it can be used to resolve boundary layers with high accuracy on globally coarse meshes. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: convection–diffusion; residual-free bubbles

1. THE ENHANCED RESIDUAL-FREE BUBBLE METHOD

The *residual-free bubble method* (RFB) was introduced by Brezzi and Russo [1] as a stable parameter-free finite element method for strongly convection-dominated convection–diffusion boundary value problems; for the *a priori* error analysis of the method applied to general second-order elliptic problems in divergence form, see Reference [2].

The RFB finite element space includes all bubble functions on a given partition $\mathscr{T}$ of the computational domain Ω, i.e. all functions with zero trace on the skeleton of the partition $\mathscr{T}$. The idea behind the RFBe method introduced in Reference [3] is to combine this property with an enrichment of the finite element space on the skeleton of the partition to obtain a method that is more stable, and is able to resolve the boundary-layer behaviour, if required.

Given $f \in L^2(\Omega)$ with $\Omega = (0,1)^2$, we consider the following model problem:

$$\begin{aligned} Lu &:= -\varepsilon\Delta u + \mathbf{a}\cdot\nabla u = f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

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where the two components of the convection field $\mathbf{a}$ are assumed to be continuously differentiable on $\overline{\Omega}$ and the diffusion parameter ε is a positive real number. Problem (1) admits a unique weak solution u in $V = H_0^1(\Omega)$.

Equation (1) models many physical phenomena. For example, we may think of u as representing the concentration of a pollutant in a fluid that is moving at velocity $\mathbf{a}$. In this case, the first-order term in Equation (1) models the convection due to the flow velocity field $\mathbf{a}$, while the second-order term represents the diffusion of u. Equation (1) is also a fundamental model problem in the realm of computational fluid dynamics, its stable and accurate solution being a crucial step in the treatment of the incompressible Navier–Stokes equations; for, despite its apparent simplicity, the convection–diffusion equation displays some of the typical difficulties associated with the numerical solution of flow problems. In particular, in the convection-dominated regime, the solution of (1) exhibits a normal boundary layer associated with the portion of the boundary $\partial\Omega_+ \subset \partial\Omega$, where $\mathbf{a}\cdot\mathbf{n} > 0$, with $\mathbf{n}$ denoting the unit outward normal vector to $\partial\Omega$. It is in the vicinity of $\partial\Omega_+$ that we aim to obtain increased accuracy over standard Galerkin and RFB finite element methods.

The (linear and bilinear) RFB finite element method consists of seeking an approximation to the weak solution in V of problem (1) from the subspace of V given by

$$V_{\mathrm{RFB}} = V_h \oplus B_h$$

where

$$V_h = \{v_h \in H_0^1(\Omega) : w_{h|_T} \in \mathscr{P}_1(T) \text{ (or } w_{h|_T} \circ F_T \in \mathscr{Q}_1(\widehat{T})) \ \forall T \in \mathscr{T}\} \tag{2}$$

$$B_h = \bigoplus_{T\in\mathscr{T}} H_0^1(T) \tag{3}$$

As regards the RFBe method, this is defined by enlarging the approximation space V_{RFB}, i.e. by considering instead the *augmented* space (see Reference [4])

$$V_{\mathrm{a}} = V_h \oplus B_h \oplus E_h$$

where the space E_h of *edge bubbles* is defined as follows.

Owing to the richness of the bubble space B_h, it suffices to fix the value of the functions belonging to E_h on the skeleton Σ of the partition. To this end, we introduce the following pieces of notation. We define as *boundary-layer region* a neighbourhood of $\partial\Omega_+$ of width $\kappa = \varepsilon \ln(1/\varepsilon)$ in the direction orthogonal to the boundary. Then, we denote by Γ_{bl} the set of all edges contained in Ω that intersect the boundary-layer region. Further, with Ω_{out} we denote the union of all elements that do not intersect the boundary-layer region.

We define an edge bubble only in association with edges that belong to Γ_{bl}. Given $\Gamma \in \Gamma_{\mathrm{bl}}$, we define the restriction e_Γ of the edge bubble $e \in E_h$ on Γ as the solution of the one-dimensional boundary value problem

$$\begin{aligned} L_\Gamma e_\Gamma &= 1 \quad \text{in } \Gamma \\ e_\Gamma &= 0 \quad \text{on } \partial\Gamma \end{aligned} \tag{4}$$

Here, we define the restriction of the differential operator L onto Γ as $L_\Gamma e_\Gamma = -\varepsilon e''_\Gamma + a_\Gamma e'_\Gamma$, where a_Γ is the projection of $\mathbf{a}$ along Γ and $(\cdot)'$ is the derivative along the edge. For any pair of adjacent elements T and T', which share Γ as a common edge, we then define e on T as the solution of $Le=0$ on T subject to $e|_{\Gamma}=e_{\Gamma}$ and $e|_{\partial T\setminus\Gamma}=0$, analogously, for T'. Finally, we extend the function e thus obtained by 0 outside $T\cup T'$ and place the resulting function into E_h. In particular, each element of E_h vanishes on Ω_{out}. This completely defines the space of edge bubbles E_h and hence the RFBe method for the solution of (1). For similar ideas of finite element spaces incorporating the flow conditions, see Reference [5 pp. 273–278], and the more recently developed FCBI method [6].

The *a priori* analysis of the method in the pre-asymptotic regime is carried out in Reference [3]. The error bounds obtained for the RFB and RFBe methods are displayed in the following theorem.

Theorem 1.1
Let $u\in H_0^1(\Omega)$ be the solution of the boundary value problem (1) assuming that $\varepsilon\in\mathbb{R}^+$, $f\in W^2_\infty(\Omega)$ and $\mathbf{a}=(a_1,a_2)\in[\mathscr{C}^1(\overline{\Omega})]^2$, with $\operatorname{div}\mathbf{a}\leqslant 0$ and $a_1,a_2\geqslant c_{\text{a}}>0$. Moreover, let $\mathscr{T}$ be a quasi-uniform axiparallel rectangular mesh. Then, as long as $h\geqslant C\kappa$ and $\varepsilon\leqslant 1/\text{e}$, the RFB solution $u_{\text{RFB}}\in V_{\text{RFB}}=V_h\oplus B_h$ satisfies

$$\varepsilon^{1/2}|u-u_{\text{RFB}}|_{1,\Omega}+h^{-1/2}\|\mathbf{a}\cdot\nabla(u-u_{\text{RFB}})\|_{-1,\Omega}\leqslant C_1\max(\varepsilon^{1/2}h^{-1/2},\varepsilon^{1/4})+C_2 \tag{5}$$

Let $u_{\text{a}}\in V_{\text{a}}=V_h\oplus B_h\oplus E_h$ be the RFBe solution. Then, under the same hypotheses as above,

$$\varepsilon^{1/2}|u-u_{\text{a}}|_{1,\Omega}+h^{-1/2}\|\mathbf{a}\cdot\nabla(u-u_{\text{a}})\|_{-1,\Omega}\leqslant C_1\max(\varepsilon^{1/2}h^{-1/2},\varepsilon^{1/4})+C_3h \tag{6}$$

The constants C_1, C_2 and C_3 are independent of h and ε, but may depend on $\mathbf{a}$.

We emphasize once again that the error bounds (5) and (6) are valid only when $h\geqslant C\kappa$ (for a quantification of the constant C see, again, Reference [3]). This is the regime of interest, since only when the mesh does not resolve the normal layers typical of convection-dominated diffusion problems is there a need for using a stabilized finite element method. Below, the *a priori* error bounds are assessed computationally by considering a problem with a known exact solution.

2. EXAMPLES

Example 1
We consider the boundary value problem

$$\begin{aligned}-\varepsilon\Delta u+u_x+u_y&=f\quad\text{in }\Omega=(0,1)^2\\ u&=0\quad\text{on }\partial\Omega\end{aligned} \tag{7}$$

with f defined in such a way that the exact solution is given by

$$u(x,y)=2(\sin x)(1-\text{e}^{-(1-x)/\varepsilon})y^2(1-\text{e}^{-(1-y)/\varepsilon})$$

To confirm the *a priori* bounds on the energy-norm error, we solve this model problem on a sequence of uniform meshes using both the RFB and the RFBe methods for $\varepsilon=10^{-2}$.

In order to implement the method, this has to be fully discretized as described in Reference [3] (see also Reference [7]). In particular, a finite number of *bubble functions* belonging to B_h has to be computed. The edge bubbles belonging to E_h have also to be computed by solving on the appropriate edge problem (4) and using the obtained function as boundary value for the computation of the corresponding edge bubble. As in Reference [8], we use finite elements for such elemental computations, on subgrids of Shishkin type; the details of this are described below.

Given the 1-D boundary value problem

$$-\varepsilon v_h'' + a v_h' = f \quad \text{in } I_h = (0, h)$$

$$v_h(0) = 0, \quad v_h(h) = 0$$

we scale it back to the unit interval, to obtain, with $\varepsilon^* = \varepsilon/h$:

$$\begin{aligned} -\varepsilon^* v'' + a v' &= hf \quad \text{in } I = (0, 1) \\ v(0) = 0, &\quad v(1) = 0 \end{aligned} \tag{8}$$

Next, we perform a finite element approximation of v on a Shishkin mesh on I: this is a piecewise uniform mesh on I consisting of N subdivisions (with N even), defined as follows. Given the *turning point* $\lambda^* = c_\mathrm{s}(\varepsilon^*/c_\mathrm{a}) \ln N$, where c_s is a constant independent of ε^* and N, the mesh is taken to be uniform with $N/2$ subdivisions on each of the subintervals $[0, 1-\lambda^*]$ and $[1-\lambda^*, 1]$. For the continuous piecewise linear finite element approximation v^I of the solution v of (8), the interpolation error over such a mesh satisfies the error bound

$$\varepsilon^* |v - v^I|_{1,I}^2 + \|v - v^I\|_{0,I}^2 \leqslant CN^{-2} \ln^2 N$$

with the constant C independent of ε and N (see Reference [5]).

Scaling back to the interval I_h we obtain a Shishkin mesh with turning point $\lambda = c_\mathrm{s}(\varepsilon/c_\mathrm{a}) \ln N$ and the scaled error bound

$$\varepsilon |v - v^I|_{1,I_h}^2 + h^{-1} \|v - v^I\|_{0,I_h}^2 \leqslant CN^{-2} \ln^2 N$$

Shishkin meshes on rectangles are constructed by taking a tensor-product of 1-D meshes, and then similar approximation results apply.

We name RFBe(N) the fully discrete RFBe method, where N is the subgrid mesh parameter. In the present example, the subgrid is the $N \times N$ axiparallel tensor-product Shishkin mesh with turning point $\lambda = c_\mathrm{s}\varepsilon/c_\mathrm{a} \ln N$ and the value of the Shishkin parameter $c_\mathrm{s} = 1$.

The convergence history in terms of the mesh parameter h is shown in the log–log plots of Figure 1; see the left-hand plots for the $\varepsilon^{1/2}$-weighted H^1-seminorm (in short, H_ε^1) errors, and those on the right for the error in the L^2-norm. To study the stabilization properties of the new method, we also plot the error in the norms restricted to the *outside region* Ω_out (see the plots below in Figure 1), which are defined by

$$|w|_{H^1_{\varepsilon,\mathrm{out}}} = \left(\frac{\varepsilon}{|\Omega_\mathrm{out}|}\right)^{1/2} |w|_{H^1(\Omega)}, \quad \|w\|_{L^2_\mathrm{out}} = \left(\frac{1}{|\Omega_\mathrm{out}|}\right)^{1/2} \|w\|_{L^2(\Omega)}$$

The error reduction rates predicted by the *a priori* analysis are confirmed by the top-left plot in Figure 1: indeed, as h decreases the RFBe(64) solution initially approaches the reference

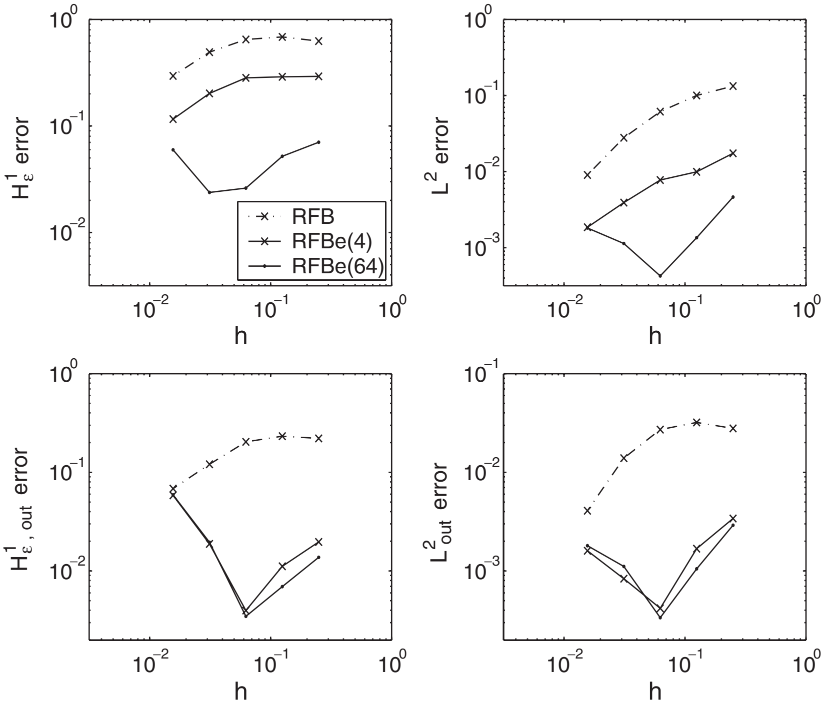

Figure 1. Example 1. $\sqrt{\varepsilon}$-weighted energy norm and L_2 norm error as a function of h; $\varepsilon = 10^{-2}$.

solution with rate 1. As we keep decreasing h, the slope of the error curve changes sign until the error curve joins the corresponding error curve for the RFB method.

The RFBe(4) method exhibits a different behaviour: since the error is concentrated in the boundary-layer region, the poor evaluation of the edge bubbles on the coarse 4×4 subgrid dominates the overall computational error. Indeed, by solving repeatedly on a fixed uniform 8×8 mesh on Ω, but using different mesh sizes N for the subgrid, we observe the characteristic $N^{-1} \log N$ convergence rate on Shishkin meshes described above, see Figure 2.

Finally, the bottom plots in Figure 1, reporting the error in Ω_{out}, show that the new method RFBe has better stability properties than RFB. Notice that, this time, there is not a significant difference between RFBe(4) and RFBe(64). We conclude that the stabilization effect due to the introduction of the edge bubbles is quite robust with respect to the accuracy of the computation of the edge bubbles.

Example 2

To exemplify the stabilization effect of the edge bubbles we solve problem (7), this time with $f = 0$ and boundary conditions $u = 1$ for $x = 0$ or $y = 0$, and a homogeneous Dirichlet boundary condition elsewhere. As we can see by comparing the solution profiles in Figure 3 (here only the bilinear part of the solution (belonging to V_h) is plotted), the edge bubbles have the effect of reducing the over- and under-shoots typical of RFB (and of most stabilized finite element methods) near the boundary layer.

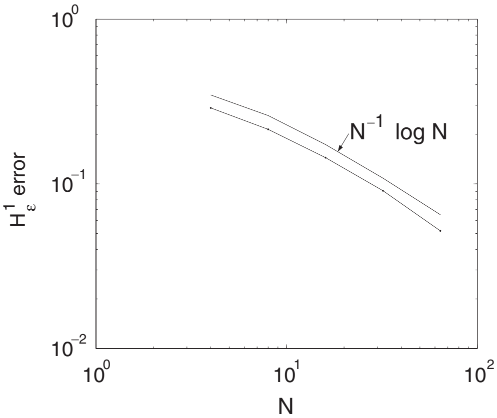

Figure 2. Example 1. $\sqrt{\varepsilon}$-weighted energy-norm error on a uniform 8×8 mesh as a function of the subgrid discretization parameter N with $\varepsilon=10^{-2}$.

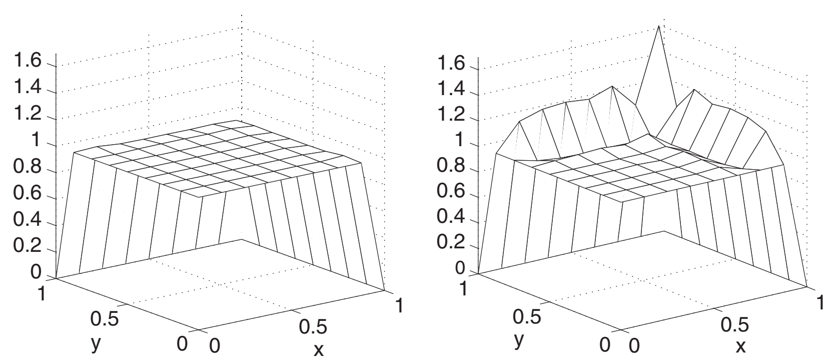

Figure 3. Example 2. Solution profiles ($\varepsilon=10^{-2}$): RFBe(4) (left) and RFB (right).

3. CONCLUSIONS

We have shown how a small number of edge bubbles can be introduced to improve the resolution of boundary layers of the RFB method in the context of elliptic convection-dominated convection–diffusion problems. The resulting scheme has better accuracy and stability properties than RFB in the regime $h \geqslant \kappa$. Although we modified the RFB scheme locally (i.e. in a neighbourhood of the boundary layer), we obtained increased resolution globally, indicating that the local introduction of the edge bubbles has a global stabilizing effect. Moreover, the method is sensitive to the accuracy of the evaluation of the edge bubbles only inasmuch as accuracy within the layer is concerned. Hence, the method can be seen either as a way of performing local refinement near the layer, or as a computationally competitive enhancement of the classical RFB method. The RFBe method can be generalized to the solution of convection–diffusion equations with a symmetric tensor diffusion coefficient, see

Reference [3]. The adaptive detection of layers where edge bubbles are required is another area of our research. Our results in this direction will be reported elsewhere.

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